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SYMMETRIC RECOLLEMENTS INDUCED BY BIMODULE EXTENSIONS

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Abstract. Inspired by the work of Jørgensen [J], we define a (upper-, lower-) symmetric recollements; and give a one-one correspondence between the equivalent classes of the upper-symmetric recollements and one of the lower-symmetric recollements, of a triangulated category. Let $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ with bimodule ${}_AM_B$. We construct an upper-symmetric abelian category recollement of Λ-mod; and a symmetric triangulated category recollement of $\underline{\Lambda\text{-}\mathcal{G}proj}$ if A and B are Gorenstein and ${}_AM$ and M_B are projective.

Key words and phrases. *abelian category, triangulated category, symmetric recollement, Gorenstein-projective modules*

Introduction

A triangulated category recollement, introduced by A. A. Beilinson, J. Bernstein, and P. Deligne [BBD], and an abelian category recollement, formulated by V. Franjou and T. Pirashvili [FV], play an important role in algebraic geometry and in representation theory ([MV], [CPS], [K], [M]).

Recently, P. Jørgensen [J] observed that if a triangulated category $\mathcal{C}$ has a Serre functor, then a triangulated category recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$ can be interchanged in two ways to triangulated category recollements of $\mathcal{C}$ relative to $\mathcal{C}''$ and $\mathcal{C}'$. Inspired by [J] we define in Section 2 a (upper-, lower-) symmetric recollement; and prove that there is a one-one correspondence between the equivalent classes of the upper-symmetric triangulated category recollements of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, and the ones of the lower-symmetric triangulated category recollements of $\mathcal{C}$ relative to $\mathcal{C}''$ and $\mathcal{C}'$. Let A and B be Artin algebras, M an A-B-bimodule, and $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ the upper triangular matrix algebra. We construct an upper-symmetric abelian category recollement of Λ-mod, the category of finitely generated Λ-modules.

An important feature of Gorenstein-projective modules is that the category A-$\mathcal{G}proj$ of Gorenstein-projective A-modules is a Frobenius category, and hence the stable category $\underline{A\text{-}\mathcal{G}proj}$ is a triangulated category ([Hap]). Iyama-Kato-Miyachi ([IKM], Theorem 3.8) prove that if A is a Gorenstein algebra, then $\underline{T_2(A)\text{-}\mathcal{G}proj}$ admits a triangulated category recollement, where $T_2(A) = \left(\begin{smallmatrix} A & A \\ 0 & A \end{smallmatrix}\right)$. In Section 3, if A and B are Gorenstein algebras and ${}_AM$ and M_B are projective, we extend this result by asserting that $\underline{\Lambda\text{-}\mathcal{G}proj}$ admits a symmetric triangulated category recollement, and by explicitly writing out the involved functors.

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1. An equivalent definition of triangulated category recollements

1.1. Recall the following

Definition 1.1. (1) *([BBD]) Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be triangulated categories. The diagram*

$$\begin{array}{ccccc} & \xleftarrow{\;i^*\;} & & \xleftarrow{\;j_!\;} & \\ \mathcal{C}' & \xrightarrow{\;i_*\;} & \mathcal{C} & \xrightarrow{\;j^*\;} & \mathcal{C}'' \\ & \xleftarrow[\;i^!\;]{} & & \xleftarrow[\;j_*\;]{} & \end{array} \tag{1.1}$$

of **exact** *functors is a triangulated category recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, if the following conditions are satisfied:*

(R1) *(i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, and (j^*, j_*) are adjoint pairs;*

(R2) *i_*, $j_!$ and j_* are fully faithful;*

(R3) *$j^*i_* = 0$;*

(R4) *For each object $X \in \mathcal{C}$, the counits and units give rise to distinguished triangles:*

$$j_!j^*X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} i_*i^*X \longrightarrow \quad \text{and} \quad i_*i^!X \xrightarrow{\omega_X} X \xrightarrow{\zeta_X} j_*j^*X \longrightarrow .$$

(2) *([FV]) Let $\mathcal{C}'$, $\mathcal{C}$ and $\mathcal{C}''$ be abelian categories. The diagram* (1.1) *of* **additive** *functors is an abelian category recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, if* (R1), (R2) *and* (R5) *are satisfied, where*

(R5) $\mathrm{Im} i_* = \mathrm{Ker} j^*$.

Remark 1.2. (1) *Let* (1.1) *be an abelian category recollement. If all the involved functors are exact, then one can prove that there is an equivalence $\mathcal{C} \cong \mathcal{C}' \times \mathcal{C}''$ of categories. This explains why Franjou-Pirashvili [FV] did not require the exactness of the involved functors in Definition 1.1*(2)*.*

(2) *For any adjoint pair (F, G), it is well-known that F is fully faithful if and only if the unit $\eta : \mathrm{Id} \to GF$ is a natural isomorphism, and G is fully faithful if and only if the counit $\epsilon : FG \longrightarrow \mathrm{Id}$ is a natural isomorphism; and that if F is fully faithful then $G\epsilon_X$ is an isomorphism for each object X, and if G is fully faithful then $F\eta_X$ is an isomorphism for each object Y.*

(3) *In any triangulated or abelian category recollement, under the condition* (R1)*, the condition* (R2) *is equivalent to the condition* (R2′)*: the units $\mathrm{Id}_{\mathcal{C}'} \to i^!i_*$ and $\mathrm{Id}_{\mathcal{C}''} \to j^*j_!$, and the counits $i^*i_* \to \mathrm{Id}_{\mathcal{C}'}$ and $j^*j_* \to \mathrm{Id}_{\mathcal{C}''}$, are natural isomorphisms.*

(4) *In an abelian category recollement one has $i^*j_! = 0$ and $i^!j_* = 0$; and in a triangulated category recollement one has $\mathrm{Im} i_* = \mathrm{Ker} j^*$, $\mathrm{Im} j^! = \mathrm{Ker} i^*$ and $\mathrm{Im} j_* = \mathrm{Ker} i^!$.*

(5) *In any abelian category recollement* (1.1)*, the counits and units give rise to exact sequences of natural transformations $j_!j^* \to \mathrm{Id}_{\mathcal{A}} \to i_*i^* \to 0$ and $0 \to i_*i^! \to \mathrm{Id}_{\mathcal{A}} \to j_*j^*$; and if $\mathcal{C}', \mathcal{C}$, and $\mathcal{C}''$ have enough projective objects, then i^* is exact if and only if $i^!j_! = 0$; and dually, if $\mathcal{C}', \mathcal{C}$, and $\mathcal{C}''$ have enough injective objects, then $i^!$ is exact if and only if $i^*j_* = 0$. See [FV].*

1.2. We will need the following equivalent definition of a triangulated category recollement, which possibly makes the construction of a triangulated category recollement easier.

Lemma 1.3. *Let* (1.1) *be a diagram of exact functors of triangulated categories. Then it is a triangulated category recollement if and only if the conditions* (R1), (R2) *and* (R5) *are satisfied.*

Proof. This seems to be well-known, however we did not find an exact reference. For the convenience of the reader we include a proof.

We only need to prove the sufficiency. Embedding the counit morphism ϵ_X into a distinguished triangle $j_!j^*X \overset{\epsilon_X}{\to} X \overset{h}{\to} Z \to$. Applying j^* we get a distinguished triangle $j^*j_!j^*X \overset{j^*\epsilon_X}{\to} j^*X \overset{j^*h}{\to} j^*Z \to$. Since $j^*\epsilon_X$ is an isomorphism by Remark 1.2(2), we have $j^*Z = 0$. By $\mathrm{Im} i_* = \mathrm{Ker} j^*$ we have $Z = i_*Z'$. Applying i^* to the distinguished triangle $j_!j^*X \overset{\epsilon_X}{\to} X \overset{h}{\to} i_*Z' \to$, by $i^*j_! = 0$ we know that $i^*h:\ i^*X \to i^*i_*Z'$ is an isomorphism. Since the counit morphism $i^*i_*Z' \xrightarrow{\varepsilon_{Z'}} Z'$ is an isomorphism, we have isomorphism $i_*((i^*h)^{-1})i_*(\varepsilon_{Z'}^{-1}):\ i_*Z' \to i_*i^*X$, and hence we get a distinguished triangle of the form $j_!j^*X \overset{\epsilon_X}{\to} X \overset{f}{\to} i_*i^*X \to$ with $f = i_*((i^*h)^{-1})i_*(\varepsilon_{Z'}^{-1})h$, which also means $\mathrm{Im} j_! = \mathrm{Ker} i^*$. Since i^*h is an isomorphism, i^*f is an isomorphism.

In order to complete the first distinguished triangle in (R4), we need to show that f can be chosen to be the unit morphism. Embedding the unit morphism η_X into a distinguished triangle $Y \to X \overset{\eta_X}{\to} i_*i^*X \to$. By the similar argument (but this time we use $\mathrm{Im} j_! = \mathrm{Ker} i^*$) we get a distinguished triangle of the form $j_!j^*X \overset{g}{\to} X \overset{\eta_X}{\to} i_*i^*X \to$. By the following commutative diagram given by the adjoint pair (i^*, i_*)

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}'}(i^*i_*i^*X, i^*X) \arrow[r, "\sim"] \arrow[d, "(i^*f,-)", "\wr"'] & \mathrm{Hom}_{\mathcal{C}}(i_*i^*X, i_*i^*X) \arrow[d, "(f,-)"] \\
\mathrm{Hom}_{\mathcal{C}'}(i^*X, i^*X) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{C}}(X, i_*i^*X)
\end{tikzcd}$$

we see that $\mathrm{Hom}_{\mathcal{C}}(f, i_*i^*X)$ is also an isomorphism, and hence there is $u \in \mathrm{Hom}_{\mathcal{C}}(i_*i^*X, i_*i^*X)$ such that $uf = \eta_X$. Since (i^*, i_*) is an adjoint pair and i_* is fully faithful, it follows that $i^*\eta_X$ is an isomorphism. Replacing f by η_X we get $v \in \mathrm{Hom}_{\mathcal{C}}(i_*i^*X, i_*i^*X)$ such that $v\eta_X = f$. Thus we have morphisms of distinguished triangles

$$\begin{tikzcd}
j_!j^*X \arrow[r, "\epsilon_X"] \arrow[d, "="] & X \arrow[r, "f"] \arrow[d, "="] & i_*i^*X \arrow[r] \arrow[d, "vu"'] & {} \\
j_!j^*X \arrow[r, "\epsilon_X"] & X \arrow[r, "f"] & i_*i^*X \arrow[r] & {}
\end{tikzcd}$$

and

$$\begin{tikzcd}
j_!j^*X \arrow[r, "g"] \arrow[d, "="] & X \arrow[r, "\eta_X"] \arrow[d, "="] & i_*i^*X \arrow[r] \arrow[d, "uv"'] & {} \\
j_!j^*X \arrow[r, "g"] & X \arrow[r, "\eta_X"] & i_*i^*X \arrow[r] & {}\ .
\end{tikzcd}$$

So uv and vu, and hence u and v, are isomorphisms. By the isomorphism of triangles

$$\begin{tikzcd} j_!j^*X \arrow[r, "\epsilon_X"] \arrow[d, "="] & X \arrow[r, "\eta_X"] \arrow[d, "="] & i_*i^*X \arrow[r] \arrow[d, "v"', "\wr"] & {} \\ j_!j^*X \arrow[r, "\epsilon_X"] & X \arrow[r, "f"] & i_*i^*X \arrow[r] & {} \end{tikzcd}$$

we see that $j_!j^*X \stackrel{\epsilon_X}{\to} X \stackrel{\eta_X}{\to} i_*i^*X \to$ is a distinguished triangle.

In order to obtain the second distinguished triangle, we embed the unit morphism ζ_X into a distinguished triangle $W \stackrel{w}{\to} X \stackrel{\zeta_X}{\to} j_*j^*X \to$. Applying j^* we get a distinguished triangle $j^*W \stackrel{j^*w}{\to} j^*X \stackrel{j^*\zeta_X}{\to} j^*j_*j^*X \to$. Since $j^*\zeta_X$ is an isomorphism by Remark 1.2(2), we have $j^*W = 0$. By $\mathrm{Im} i_* = \mathrm{Ker} j^*$ we have $W = i_*X'$. Applying $i^!$ to the distinguished triangle $i_*X' \stackrel{w}{\to} X \stackrel{\zeta_X}{\to} j_*j^*X \to$ and by $i^!j_* = 0$ we know that $i^!w: \ i^!i_*X' \to i^!X$ is an isomorphism. Using the unit isomorphism $X' \to i^!i_*X'$, we get a distinguished triangle of the form $i_*i^!X \stackrel{a}{\to} X \stackrel{\zeta_X}{\to} j_*j^*X \to$ with $i^!a$ an isomorphism. It follows that $\mathrm{Im} j_* = \mathrm{Ker} i^!$.

Now since $\mathrm{Im} j_* = \mathrm{Ker} i^!$ and $\mathrm{Im} i_* = \mathrm{Ker} j^*$, it follows that we can replace (i^*, i_*) by (j^*, j_*), and replace $(j_!, j^*)$ by $(i_*, i^!)$, in the distinguished triangle $j_!j^*X \stackrel{\epsilon_X}{\to} X \stackrel{\eta_X}{\to} i_*i^*X \to$. In this way we get the second distinguished triangle $i_*i^!X \stackrel{\omega_X}{\to} X \stackrel{\zeta_X}{\to} j_*j^*X \to$. ■

2. Upper-symmetric recollements

2.1. Given a recollement of $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$, one usually can **not** expect a recollement of $\mathcal{C}$ relative to $\mathcal{C}''$ and $\mathcal{C}'$. Inspired by [J] we define

Definition 2.1. *([J]) A triangulated category recollement*

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{C}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \quad (2.1)$$

of $\mathcal{C}$ is upper-symmetric, if there are exact functors $j_?$ and $i_?$ such that

$$\begin{tikzcd} \mathcal{C}'' \arrow[r, "j_*"] & \mathcal{C} \arrow[l, bend right, "j^*"'] \arrow[l, bend left, "j_?"] \arrow[r, "i^!"] & \mathcal{C}' \arrow[l, bend right, "i_*"'] \arrow[l, bend left, "i_?"] \end{tikzcd} \quad (2.2)$$

is a recollement; and it is lower-symmetric, if there are exact functors $j^?$ and $i^?$ such that

$$\begin{tikzcd} \mathcal{C}'' \arrow[r, "j_!"] & \mathcal{C} \arrow[l, bend right, "j^?"'] \arrow[l, bend left, "j^*"] \arrow[r, "i^*"] & \mathcal{C}' \arrow[l, bend right, "i^?"'] \arrow[l, bend left, "i_*"] \end{tikzcd} \quad (2.3)$$

is a recollement. A recollement is symmetric if it is upper- and lower-symmetric.

Similarly, we have a (upper-, lower-) symmetric abelian category recollement, and note that in abelian situations, all the involved functors, in particular $j_?$, $i_?$, $j^?$ and $i^?$, are only required to be **additive** *functors,* **not** *required to be exact.*

Let k be a field. P. Jørgensen [J] observed that if a Hom-finite k-linear triangulated category $\mathcal{C}$ has a Serre functor, then any recollement of $\mathcal{C}$ is symmetric: his proof does not use any triangulated structure of $\mathcal{C}$ and hence also works for a Hom-finite k-linear abelian category having a Serre functor. For a similar notion of symmetric recollements of unbounded derived categories we refer to S. König [K], and also Chen-Lin [CL].

2.2. Given two triangulated or abelian category recollements

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_*"] & \mathcal{C} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{C}'' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C}' \arrow[r, "i_*^D"] & \mathcal{D} \arrow[l, bend right, "i_D^*"'] \arrow[l, bend left, "i_D^!"] \arrow[r, "j_D^*"] & \mathcal{C}'' \arrow[l, bend right, "j_!^D"'] \arrow[l, bend left, "j_*^D"] \end{tikzcd}$$

if there is an exact functor $f : \mathcal{C} \to \mathcal{D}$ such that there are natural isomorphisms

$$i^* \approx i_D^* f, \quad f i_* \approx i_*^D, \quad i^! \approx i_D^! f, \quad f j_! \approx j_!^D, \quad j^* \approx j_D^* f, \quad f j_* \approx j_*^D,$$

then we call f *a comparison functor*. Two (triangulated or abelian category) recollements are *equivalent* if there is a comparison functor f which is an equivalence of categories. According to Parshall-Scott [PS, Theorem 2.5], a comparison functor between triangulated category recollements is an equivalence of categories. However, Franjou-Pirashvili [FV] pointed out that this is not necessarily the case for abelian category recollements.

2.3. In this subsection we only consider triangulated category recollements. If (2.1) is an upper-symmetric recollement, then we call (2.2) *a upper-symmetric version* of (2.1); and if (2.1) is an lower-symmetric recollement, then we call (2.3) *a lower-symmetric version* of (2.1).

Lemma 2.2. (1) *Any two upper-symmetric versions of a upper-symmetric recollement are equivalent.*

(1′) *Any two lower-symmetric versions of a lower-symmetric recollement are equivalent.*

(2) *Equivalent upper-symmetric recollements have equivalent upper-symmetric versions.*

(2′) *Equivalent lower-symmetric recollements have equivalent lower-symmetric versions.*

Proof. (1) Let (2.2) and

$$\begin{tikzcd} \mathcal{C}'' \arrow[r, "j_*"] & \mathcal{C} \arrow[l, bend right, "j^*"'] \arrow[l, bend left, "j_{??}"] \arrow[r, "i^!"] & \mathcal{C}' \arrow[l, bend right, "i_*"'] \arrow[l, bend left, "i_{??}"] \end{tikzcd} \tag{2.4}$$

be two upper-symmetric versions of a upper-symmetric recollement (2.1). Then $j_{??}i_? = 0$. In fact, for $Y \in \mathcal{C}'$ we have

$$\mathrm{Hom}_{\mathcal{C}''}(j_{??}i_?Y, j_{??}i_?Y) \cong \mathrm{Hom}_{\mathcal{C}}(j_*j_{??}i_?Y, i_?Y) \cong \mathrm{Hom}_{\mathcal{C}''}(i^!j_*j_{??}i_?Y, Y) = 0.$$

For $X \in \mathcal{C}$, by (2.2) and (R4) we have distinguished triangle $j_*j_?X \xrightarrow{\epsilon_X} X \xrightarrow{\eta_X} i_?i^!X \to$. Applying exact functor $j_{??}$ and using the unit $\mathrm{Id}_{\mathcal{C}''} \to j_{??}j_*$, we have

$$j_{??}X \cong j_{??}j_*j_?X \cong j_?X,$$

which means that $j_{??}$ is naturally isomorphic to $j_?$. Similarly one can prove that $i_{??}$ is naturally isomorphic to $i_?$. Thus $\mathrm{Id}_{\mathcal{C}}$ is an equivalence between (2.2) and (2.4). This proves (1).

$(1')$ can be similarly proved.

(2) Given two equivalent upper-symmetric recollements

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "i_*"] & \mathcal{C} \arrow[l, shift right=6, "i^*"'] \arrow[l, shift left=6, "i^!"'] \arrow[r, "j^*"] & \mathcal{C}'' \arrow[l, shift right=6, "j_!"'] \arrow[l, shift left=6, "j_*"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{C}' \arrow[r, "i_*^D"] & \mathcal{D} \arrow[l, shift right=6, "i_D^*"'] \arrow[l, shift left=6, "i_D^!"'] \arrow[r, "j_D^*"] & \mathcal{C}'' \arrow[l, shift right=6, "j_!^D"'] \arrow[l, shift left=6, "j_*^D"'] \end{tikzcd}$$

with comparison functor f, let (2.2) as an upper-symmetric version of the first recollement. By Lemma 1.3 we know that

$$\begin{tikzcd} \mathcal{C}'' \arrow[r, "j_*^D"] & \mathcal{D} \arrow[l, shift right=6, "j_D^*"'] \arrow[l, shift left=6, "j_? f^{-1}"'] \arrow[r, "i_D^!"] & \mathcal{C}' \arrow[l, shift right=6, "i_*^D"'] \arrow[l, shift left=6, "f i_?"'] \end{tikzcd} \cdot \tag{2.5}$$

is a triangulated category recollement, and that f is an equivalence between (2.2) and (2.5). Note that (2.5) is an upper-symmetric version of the second given upper-symmetric recollement, and hence the assertion follows from (1).

$(2')$ can be similarly proved. ■

Let $\mathcal{C}', \mathcal{C}, \mathcal{C}''$ be triangulated categories. Denote by $USR(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ the class of equivalence classes of the upper-symmetric recollements of triangulated category $\mathcal{C}$ relative to $\mathcal{C}'$ and $\mathcal{C}''$; and denote by $LSR(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ the class of the lower-symmetric recollements of triangulated category $\mathcal{C}$ relative to $\mathcal{C}''$ and $\mathcal{C}'$.

Theorem 2.3. *There is a one-one correspondence between $USR(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ and $LSR(\mathcal{C}'', \mathcal{C}, \mathcal{C}')$.*

Proof. Given an upper-symmetric recollement (2.1), observe that an upper-symmetric version (2.2) of (2.1) is lower-symmetric: in fact, (2.1) could be a lower-symmetric version of (2.2). Similarly, a lower-symmetric recollement could be a upper-symmetric version of a lower-symmetric version of itself. Thus by Lemma 2.2 we get a one-one correspondence between $USR(\mathcal{C}', \mathcal{C}, \mathcal{C}'')$ and $LSR(\mathcal{C}'', \mathcal{C}, \mathcal{C}')$. ■

2.4. We consider Artin algebras over a fixed commutative artinian ring, and finitely generated modules. Let A and B be Artin algebras, and M an A-B-bimodule. Then $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$ is an Artin algebra with multiplication given by the one of matrices. Denoted by A-mod the category of finitely generated left A-modules. A left Λ-module is identified with a triple $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi$, or simply $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)$ if ϕ is clear, where $X \in A$-mod, $Y \in B$-mod, and $\phi : M \otimes_B Y \to X$ is an A-map. A Λ-map $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \to \left(\begin{smallmatrix} X' \\ Y' \end{smallmatrix}\right)_{\phi'}$ is identified with a pair $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right)$, where $f \in \mathrm{Hom}_A(X, X')$, $g \in \mathrm{Hom}_B(Y, Y')$, such that $\phi'(\mathrm{Id} \otimes g) = f\phi$. The indecomposable projective Λ-modules are exactly $\left(\begin{smallmatrix} P \\ 0 \end{smallmatrix}\right)$ and $\left(\begin{smallmatrix} M \otimes_B Q \\ Q \end{smallmatrix}\right)_{\mathrm{id}}$, where P runs over indecomposable projective A-modules, and Q runs over indecomposable projective B-modules. See [ARS], p.73.

For any A-module X and B-module Y, denote by $\alpha_{X,Y}$ the adjoint isomorphism

$$\alpha_{X,Y} : \mathrm{Hom}_A(M \otimes_B Y, X) \longrightarrow \mathrm{Hom}_B(Y, \mathrm{Hom}_A(M, X))$$

given by

$$\alpha_{X,Y}(\phi)(y)(m) = \phi(m \otimes y), \ \forall \ \phi \in \mathrm{Hom}_A(M \otimes_B Y, X), \ y \in Y, \ m \in M.$$

Put ψ_X to be $\alpha^{-1}_{X,\mathrm{Hom}(M,X)}(\mathrm{Id}_{\mathrm{Hom}(M,X)})$. Thus $\psi_X : M \otimes_B \mathrm{Hom}_A(M,X) \to X$ is given by $m \otimes f \mapsto f(m)$.

Theorem 2.4. *Let A and B be Artin algebras, ${}_AM_B$ an A-B-bimodule, and $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$. Then we have an upper-symmetric (but non lower-symmetric) abelian category recollement*

$$\begin{tikzcd} A\text{-mod} \arrow[r, "i_*"] & \Lambda\text{-mod} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & B\text{-mod} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd} \tag{2.6}$$

where

i^* *is given by* $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \mapsto \mathrm{Coker}\phi$; i_* *is given by* $X \mapsto \left(\begin{smallmatrix} X \\ 0 \end{smallmatrix}\right)$; $i^!$ *is given by* $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \mapsto X$;

$j_!$ *is given by* $Y \mapsto \left(\begin{smallmatrix} M\otimes Y \\ Y \end{smallmatrix}\right)_{\mathrm{Id}}$; j^* *is given by* $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \mapsto Y$; j_* *is given by* $Y \mapsto \left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right)$;

$j_?$ *is given by* $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \mapsto \mathrm{Ker}\,\alpha_{X,Y}(\phi)$; *and* $i_?$ *is given by* $X \mapsto \left(\begin{smallmatrix} X \\ \mathrm{Hom}_A(M,X) \end{smallmatrix}\right)_{\psi_X}$.

Proof. By construction i_*, $j_!$ and j_* are fully faithful; $\mathrm{Im}i_* = \mathrm{Ker}j^*$, and $\mathrm{Im}j_* = \mathrm{Ker}i^!$. For $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \in \Lambda$-mod, $X' \in A$-mod, and $Y' \in B$-mod, we have the following isomorphisms of abelian groups, which are natural in both positions

$$\mathrm{Hom}_A(\mathrm{Coker}\,\phi, X') \cong \mathrm{Hom}_\Lambda(\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi, \left(\begin{smallmatrix} X' \\ 0 \end{smallmatrix}\right)) \tag{2.7}$$

given by $f \mapsto \left(\begin{smallmatrix} f\pi \\ 0 \end{smallmatrix}\right)$, where $\pi : X \to \mathrm{Coker}\,\phi$ is the canonical A-map;

$$\mathrm{Hom}_\Lambda(\left(\begin{smallmatrix} X' \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi) \cong \mathrm{Hom}_A(X', X); \tag{2.8}$$

$$\mathrm{Hom}_\Lambda(\left(\begin{smallmatrix} M\otimes Y' \\ Y' \end{smallmatrix}\right)_{\mathrm{Id}}, \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi) \cong \mathrm{Hom}_B(Y', Y) \tag{2.9}$$

given by $\left(\begin{smallmatrix} \phi(\mathrm{Id}\otimes g) \\ g \end{smallmatrix}\right) \mapsto g$; and

$$\mathrm{Hom}_B(Y, Y') \cong \mathrm{Hom}_\Lambda(\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi, \left(\begin{smallmatrix} 0 \\ Y' \end{smallmatrix}\right)).$$

Thus (i^*, i_*), $(i_*, i^!)$, $(j_!, j^*)$, and (j^*, j_*) are adjoint pairs, and hence (2.6) is a recollement. It is not lower-symmetric since $\mathrm{Im}j_! \neq \mathrm{Ker}i^*$.

In order to see that it is upper-symmetric, it remains to prove that $(j_*, j_?)$ and $(i^!, i_?)$ are adjoint pairs, and that $i_?$ is fully faithful. For $g \in \mathrm{Hom}_B(Y, Y')$ and $\left(\begin{smallmatrix} X' \\ Y' \end{smallmatrix}\right)_{\phi'} \in \Lambda$-mod, we have

$$\left(\begin{smallmatrix} 0 \\ g \end{smallmatrix}\right) \in \mathrm{Hom}_\Lambda(\left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right), \left(\begin{smallmatrix} X' \\ Y' \end{smallmatrix}\right)_{\phi'}) \Longleftrightarrow \phi'(\mathrm{Id}\otimes g) = 0 \Longleftrightarrow \phi'(m \otimes g(y)) = 0, \ \forall \ y \in Y, \ \forall \ m \in M$$

$$\Longleftrightarrow \alpha_{X',Y'}(\phi')(g(y)) = 0, \ \forall \ y \in Y \Longleftrightarrow g(Y) \subseteq \mathrm{Ker}\,\alpha_{X',Y'}(\phi') \Longleftrightarrow g \in \mathrm{Hom}_B(Y, \mathrm{Ker}\,\alpha_{X',Y'}(\phi')).$$

It follows that $\left(\begin{smallmatrix} 0 \\ g \end{smallmatrix}\right) \mapsto g$ gives an isomorphism $\mathrm{Hom}_\Lambda(\left(\begin{smallmatrix} 0 \\ Y \end{smallmatrix}\right), \left(\begin{smallmatrix} X' \\ Y' \end{smallmatrix}\right)_{\phi'}) \to \mathrm{Hom}_B(Y, \mathrm{Ker}\,\alpha_{X',Y'}(\phi'))$ of abelian groups, which is natural in both positions, i.e., $(j_*, j_?)$ is an adjoint pair. Let $\left(\begin{smallmatrix} f \\ g \end{smallmatrix}\right) \in \mathrm{Hom}_\Lambda(\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi, \left(\begin{smallmatrix} X' \\ \mathrm{Hom}_A(M,X') \end{smallmatrix}\right)_{\psi_{X'}})$. By $\psi_{X'}(\mathrm{id}\otimes g) = f\phi$ we have

$$\begin{aligned} \alpha_{X',Y}(f\phi)(y)(m) &= f\phi(m \otimes y) = \psi_{X'}(\mathrm{Id}\otimes g)(m \otimes y) \\ &= \psi_{X'}(m \otimes g(y)) = g(y)(m), \ \forall \ y \in Y, \ \forall \ m \in M, \end{aligned}$$

which means $g = \alpha_{_{X',Y}}(f\phi)$. Thus $f \mapsto \left(\begin{smallmatrix} f \\ \alpha_{_{X',Y}}(f\phi)\end{smallmatrix}\right)$ gives an isomorphism

$$\operatorname{Hom}_A(X, X') \longrightarrow \operatorname{Hom}_\Lambda(\left(\begin{smallmatrix} X \\ Y\end{smallmatrix}\right)_\phi, \left(\begin{smallmatrix} X' \\ \operatorname{Hom}_A(M,X')\end{smallmatrix}\right)_{\psi_{X'}})$$

of abelian groups, which is natural in both positions, i.e., $(i^!, i_?)$ is an adjoint pair. Since $\alpha_{_{X',\operatorname{Hom}(M,X)}}(f\psi_{_X}) = \operatorname{Hom}_A(M, f)$, this isomorphism also shows that $i_?$ is fully faithful. This completes the proof. ■

By Theorem 2.4 we have

Corollary 2.5. *Let A be a Gorenstein algebra, and $T_2(A) = \left(\begin{smallmatrix} A & A \\ 0 & A\end{smallmatrix}\right)$. Then we have an upper-symmetric (but non lower-symmetric) abelian category recollement*

$$\begin{tikzcd} A\text{-mod} \arrow[r] & T_2(A)\text{-mod} \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \arrow[r] & A\text{-mod}. \arrow[l, shift right=2ex] \arrow[l, shift left=2ex] \end{tikzcd}$$

Remark 2.6. *As we see from* (2.6) *and its upper symmetric version, in an abelian category recollement, the following statement may* **not** *true:*

(1) $\operatorname{Im} j_! = \operatorname{Ker} i^*$*;* $\operatorname{Im} j_* = \operatorname{Ker} i^!$*;*

(2) *The counits and units give rise to exact sequences of natural transformations:*

$$0 \longrightarrow j_!j^* \longrightarrow \mathrm{Id}_\mathcal{C} \longrightarrow i_*i^* \longrightarrow 0 \quad \textit{and} \quad 0 \longrightarrow i_*i^! \longrightarrow \mathrm{Id}_\mathcal{C} \longrightarrow j_*j^* \longrightarrow 0.$$

(3) $i^!j_! = 0$*; and* $i^*j_* = 0$.

In triangulated situations, (1) *and the corresponding version of* (2) **always** *hold; but* (3) *is also* **not** *true in general.*

3. Symmetric recollements induced by Gorenstein-projective modules

3.1. Let A be an Artin algebra. An A-module G is *Gorenstein-projective*, if there is an exact sequence $\cdots \to P^{-1} \to P^0 \stackrel{d^0}{\to} P^1 \to \cdots$ of projective A-modules, which stays exact under $\operatorname{Hom}_A(-, A)$, and such that $G \cong \operatorname{Ker} d^0$. Let A-$\mathcal{G}proj$ be the full subcategory of A-mod consisting of the Gorenstein-projective modules. Then A-$\mathcal{G}proj \subseteq {}^\perp A$, where ${}^\perp A = \{X \in A\text{-mod} \mid \operatorname{Ext}^i_A(X, A) = 0, \forall\ i \geq 1\}$; and $\operatorname{Hom}_A(-, {}_AA)$ induces a duality A-$\mathcal{G}proj \cong A^{op}$-$\mathcal{G}proj$ with a quasi-inverse $\operatorname{Hom}_A(-, A_A)$ ([B], Proposition 3.4). An important feature is that A-$\mathcal{G}proj$ is a Frobenius category with projective-injective objects being projective A-modules, and hence the stable category $\underline{A\text{-}\mathcal{G}proj}$ modulo projective A-modules is a triangulated category ([Hap]).

An Artin algebra A is *Gorenstein*, if inj.dim ${}_AA < \infty$ and inj.dim $A_A < \infty$. We have the following well-known fact (E. Enochs - O. Jenda [EJ], Corollary 11.5.3).

Lemma 3.1. *Let A be a Gorenstein algebra. Then*

(1) *If $P^\bullet$ is an exact sequence of projective left (resp. right) A-modules, then* $\operatorname{Hom}_A(P^\bullet,\ A)$ *is again an exact sequence of projective right (resp. left) A-modules.*

(2) *A module G is Gorenstein-projective if and only if there is an exact sequence* $0 \to G \to P^0 \to P^1 \to \cdots$ *with each P^i projective.*

(3) A-$\mathcal{G}proj = {}^{\perp}A$.

Proof. For convenience we include an alternating proof.

(1) Let $0 \to K \to I_0 \to I_1 \to 0$ be an exact sequence with $I_0,\ I_1$ injective modules. Then $0 \to \operatorname{Hom}_A(P^\bullet,\ K) \to \operatorname{Hom}_A(P^\bullet,\ I_0) \to \operatorname{Hom}_A(P^\bullet,\ I_1) \to 0$ is an exact sequence of complexes. Since $\operatorname{Hom}_A(P^\bullet,\ I_i)$ $(i = 0, 1)$ are exact, it follows that $\operatorname{Hom}_A(P^\bullet,\ K)$ is exact. Repeating this process, by inj.dim ${}_AA < \infty$ we deduce that $\operatorname{Hom}_A(P^\bullet,\ A)$ is exact.

(2) This follows from definition and (1).

(3) Let $G \in {}^{\perp}A$. Applying $\operatorname{Hom}_A(-, A)$ to a projective resolution of G we get is an exact sequence. By (2) this means that $\operatorname{Hom}_A(G, A)$ is a Gorenstein-projective right A-module, and hence G is Gorenstein-projective by the duality $\operatorname{Hom}_A(-,\ {}_AA) :\ A\text{-}\mathcal{G}proj \cong A^{op}\text{-}\mathcal{G}proj$. ■

We need the following description of Gorenstein-projective Λ-modules.

Proposition 3.2. *Let A and B be Gorenstein algebras, M an A-B-bimodule such that ${}_AM$ and M_B are projective, and $\Lambda = \left(\begin{smallmatrix} A & M \\ 0 & B \end{smallmatrix}\right)$. Then $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi$ is a Gorenstein-projective Λ-module if and only if $\phi : M \otimes Y \to X$ is monic, X and $\operatorname{Coker}\phi$ are Gorenstein-projective A-modules, and Y is a Gorenstein-projective B-module. In this case $M \otimes Y$ is a Gorenstein-projective A-module.*

Proof. If $\left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi$ is a Gorenstein-projective Λ-module, then there is an exact sequence

$$0 \longrightarrow \left(\begin{smallmatrix} X \\ Y \end{smallmatrix}\right)_\phi \longrightarrow \begin{pmatrix} P_0 \oplus (M \otimes Q_0) \\ Q_0 \end{pmatrix}_{\left(\begin{smallmatrix} 0 \\ \mathrm{Id} \end{smallmatrix}\right)} \longrightarrow \begin{pmatrix} P_1 \oplus (M \otimes Q_1) \\ Q_1 \end{pmatrix}_{\left(\begin{smallmatrix} 0 \\ \mathrm{Id} \end{smallmatrix}\right)} \longrightarrow \cdots \tag{3.1}$$

where P_i and Q_i are respectively projective A- and B-modules, $i \geq 0$, i.e., we have exact sequences

$$0 \longrightarrow X \longrightarrow P_0 \oplus (M \otimes Q_0) \longrightarrow P_1 \oplus (M \otimes Q_1) \longrightarrow \cdots \tag{3.2}$$

and

$$0 \longrightarrow Y \longrightarrow Q_0 \longrightarrow Q_1 \longrightarrow \cdots, \tag{3.3}$$

such that the following diagram commutes

$$\begin{tikzcd} 0 \arrow[r] & M \otimes_B Y \arrow[r] \arrow[d, "\phi"] & M \otimes_B Q_0 \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 \\ \mathrm{Id} \end{smallmatrix}\right)"'] & M \otimes_B Q_1 \arrow[r] \arrow[d, "\left(\begin{smallmatrix} 0 \\ \mathrm{Id} \end{smallmatrix}\right)"] & \cdots \\ 0 \arrow[r] & X \arrow[r] & P_0 \oplus (M \otimes Q_0) \arrow[r] & P_1 \oplus (M \otimes Q_1) \arrow[r] & \cdots . \end{tikzcd} \tag{3.4}$$

By Lemma 3.1(2) Y is Gorenstein-projective. Since ${}_AM$ and ${}_BQ_i$ are projective, it follows that $M \otimes Q_i$ are projective A-modules, and hence X is Gorenstein-projective by Lemma 3.1(2). Since M_B is projective, by (3.3) the upper row of (3.4) is exact, and hence $M \otimes Y$ is Gorenstein-projective and ϕ is monic. By (3.4) we get exact sequence $0 \to \operatorname{Coker}\phi \to P_0 \to P_1 \to \cdots$, thus $\operatorname{Coker}\phi$ is Gorenstein-projective by Lemma 3.1(2).

Conversely, we have exact sequence (3.3) with Q_i being projective B-modules. Since M_B is projective and $\operatorname{Coker}\phi$ is Gorenstein-projective, we get the following exact sequences

$$\begin{array}{l} 0 \longrightarrow M\otimes Y \longrightarrow M\otimes Q_0 \longrightarrow M\otimes Q_1 \longrightarrow \cdots \\ 0 \longrightarrow \operatorname{Coker}\phi \longrightarrow \quad P_0 \quad \longrightarrow \quad P_1 \quad \longrightarrow \cdots \end{array}$$

with P_i projective. Since $M\otimes Q_i$ $(i\ge 0)$ are projective A-modules and projective A-modules are injective objects in A-$\mathcal{G}proj$, it follows from the exact sequence $0\to M\otimes Y\to X\to \operatorname{Coker}\phi\to 0$ and a version of Horseshoe Lemma that there is an exact sequence (3.2) such that the diagram (3.4) commutes. This means that (3.1) is exact. Since Λ is also Gorenstein (see e.g. [C], Theorem 3.3), it follows from Lemma 3.1(2) that $\binom{X}{Y}_\phi$ is a Gorenstein-projective Λ-module. ■

3.2. The main result of this section is as follows.

Theorem 3.3. *Let A and B be Gorenstein algebras, M an A-B-bimodule such that ${}_AM$ and M_B are projective, and $\Lambda=\left(\begin{smallmatrix} A & M\\ 0 & B\end{smallmatrix}\right)$. Then we have a triangulated category recollement*

$$\begin{tikzcd} \underline{A\text{-}\mathcal{G}proj} \arrow[r, "i_*"] & \underline{\Lambda\text{-}\mathcal{G}proj} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \underline{B\text{-}\mathcal{G}proj} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}.$$

Moreover, if A and B are in additional finite-dimensional algebras over a field, then it is a symmetric recollement.

3.3. Before giving a proof, we construct all the functors in Theorem 3.3. If a Λ-map $\binom{X}{Y}_\phi\to\binom{X'}{Y'}_{\phi'}$ factors through a projective Λ-module $\binom{P}{0}\oplus\binom{M\otimes Q}{Q}$, then it is easy to see that the induced A-map $\operatorname{Coker}\phi\to\operatorname{Coker}\phi'$ factors through P. By Proposition 3.2 this implies that the functor Λ-$\mathcal{G}proj\to \underline{A\text{-}\mathcal{G}proj}$ given by $\binom{X}{Y}_\phi\mapsto\operatorname{Coker}\phi$ induces a functor $i^*:\ \underline{\Lambda\text{-}\mathcal{G}proj}\to\underline{A\text{-}\mathcal{G}proj}$.

By Proposition 3.2 there is a unique functor $i_*:\underline{A\text{-}\mathcal{G}proj}\to\underline{\Lambda\text{-}\mathcal{G}proj}$ given by $X\mapsto\binom{X}{0}$, which is fully faithful.

If a Λ-map $\binom{f}{g}:\binom{X}{Y}_\phi\to\binom{X'}{Y'}_{\phi'}$ factors through a projective Λ-module $\binom{P}{0}\oplus\binom{M\otimes Q}{Q}$, then $f:\ X\to X'$ factors through a projective A-module $P\oplus(M\otimes Q)$. By Proposition 3.2 this implies that there is a unique functor $i^!:\underline{\Lambda\text{-}\mathcal{G}proj}\to\underline{A\text{-}\mathcal{G}proj}$ given by $\binom{X}{Y}_\phi\mapsto X$.

By Proposition 3.2 there is a unique functor $j^*:\ \underline{\Lambda\text{-}\mathcal{G}proj}\to\underline{B\text{-}\mathcal{G}proj}$ given by $\binom{X}{Y}_\phi\mapsto Y$.

Let ${}_BY$ be a Gorenstein-projective module. Since M_B is projective, by Lemma 3.1(2) $M\otimes Y$ is a Gorenstein-projective A-module. By Proposition 3.2 there is a unique functor $j_!:\ \underline{B\text{-}\mathcal{G}proj}\to\underline{\Lambda\text{-}\mathcal{G}proj}$ given by $Y\mapsto\binom{M\otimes Y}{Y}_{\mathrm{Id}}$, which is fully faithful.

Lemma 3.4. *Let A, B, M, and Λ be as in Theorem 3.3. Then there exists a unique fully faithful functor $j_*:\ \underline{B\text{-}\mathcal{G}proj}\to\underline{\Lambda\text{-}\mathcal{G}proj}$ given by $Y\mapsto\binom{P}{Y}_\sigma$, where P is a projective A-module such that there is an exact sequence $0\to M\otimes Y\xrightarrow{\sigma}P\to\operatorname{Coker}\sigma\to 0$ with $\operatorname{Coker}\sigma\in A\text{-}\mathcal{G}proj$.*

Proof. Let ${}_BY$ be Gorenstein-projective. Then $M\otimes Y$ is Gorenstein-projective, and hence there is an exact sequence $0\to M\otimes Y\xrightarrow{\sigma}P\to\operatorname{Coker}\sigma\to 0$ with P projective and $\operatorname{Coker}\sigma\in A\text{-}\mathcal{G}proj$. Let $g:Y\to Y'$ be a B-map with $Y,\ Y'\in B\text{-}\mathcal{G}proj$, and P' a projective A-module such that

$0 \to M\otimes Y' \xrightarrow{\sigma'} P' \to \mathrm{Coker}\sigma' \to 0$ is exact with $\mathrm{Coker}\sigma' \in A\text{-}\mathcal{G}proj$. Since projective A-modules are injective objects in A-$\mathcal{G}proj$, it follows that there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & M\otimes Y \arrow[r, "\sigma"] \arrow[d, "1\otimes g"'] & P \arrow[r, "\pi"] \arrow[d, "f"'] & \mathrm{Coker}\sigma \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M\otimes Y' \arrow[r, "\sigma'"] & P' \arrow[r] & \mathrm{Coker}\sigma' \arrow[r] & 0.
\end{tikzcd}$$

Taking $g = \mathrm{Id}$ we see $\binom{P}{Y}_\sigma \cong \binom{P'}{Y}_{\sigma'}$ in $\underline{\Lambda\text{-}\mathcal{G}proj}$. If we have another map $f' : P \to P'$ such that $f'\sigma = \sigma'(1\otimes g)$, then $f - f'$ factors through $\mathrm{Coker}\,\sigma$. Since $\mathrm{Coker}\sigma \in A\text{-}\mathcal{G}proj$, we have a monomorphism $\tilde{\sigma} : \mathrm{Coker}\sigma \to \tilde{P}$ with $\tilde{P}$ projective. Then we easily see that $\binom{f}{g} - \binom{f'}{g}$ factors through projective Λ-module $\binom{\tilde{P}}{0}$, and hence $\underline{\binom{f}{g}} = \underline{\binom{f'}{g}}$. Thus we get a unique functor $j_* : B\text{-}\mathcal{G}proj \to \underline{\Lambda\text{-}\mathcal{G}proj}$ given by $Y \mapsto \binom{P}{Y}_\sigma$ and $g \mapsto \underline{\binom{f}{g}}$.

Assume that $g : Y \to Y'$ factors through a projective module ${}_BQ$ with $g = g_2g_1$. Since $M\otimes Q$ is projective and hence an injective object in $\underline{A\text{-}\mathcal{G}proj}$, there is an A-map $\alpha : P \to M\otimes Q$ such that $1\otimes g_1 = \alpha\sigma$. Since $(f - \sigma'(1\otimes g_2)\alpha)\sigma = 0$, there is an A-map $\tilde{f} : \mathrm{Coker}\sigma \to P'$ such that $\tilde{f}\pi = f - \sigma'(1\otimes g_2)\alpha$. Let $\tilde{\sigma} : \mathrm{Coker}\sigma \to \tilde{P}$ be a monomorphism with $\tilde{P}$ projective. Then we get an A-map $\beta : \tilde{P} \to P'$ such that $\tilde{f} = \beta\tilde{\sigma}$. Thus $\underline{\binom{f}{g}}$ factors through projective Λ-module $\binom{M\otimes Q}{Q} \oplus \binom{\tilde{P}}{0}$ with $\underline{\binom{f}{g}} = \underline{\binom{(\sigma'(1\otimes g_2),\beta)}{g_2}}\ \underline{\binom{\binom{\alpha}{\tilde{\sigma}\pi}}{g_1}}$. Therefore $j_* : B\text{-}\mathcal{G}proj \to \underline{\Lambda\text{-}\mathcal{G}proj}$ induces a functor $B\text{-}\mathcal{G}proj \to \underline{\Lambda\text{-}\mathcal{G}proj}$, again denoted by j_*, which is given by $Y \mapsto \binom{P}{Y}_\sigma$ and $\underline{g} \mapsto \underline{\binom{f}{g}}$.

By the above argument we know that j_* is full. If $\binom{f}{g}$ factors through projective Λ-module $\binom{M\otimes Q}{Q} \oplus \binom{\tilde{P}}{0}$, then g factors through projective module ${}_BQ$. Thus j_* is faithful. ∎

3.4. Let $\mathcal{A}$ be a Frobenius category and $\underline{\mathcal{A}}$ the corresponding stable category. Then $\underline{\mathcal{A}}$ is a triangulated category with shift functor [1] given by $X[1] = \mathrm{Coker}(X \longrightarrow I(X))$, where $I(X)$ is a projective-injective object of $\mathcal{A}$; each exact sequence $0 \to X \xrightarrow{u} Y \xrightarrow{v} Z \to 0$ in $\mathcal{A}$ gives rise to a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \to$ in $\underline{\mathcal{A}}$, and each distinguished triangle in $\underline{\mathcal{A}}$ is of this form up to an isomorphism. See D. Happel [H], Chapter 1, Section 2. It follows that we have

Lemma 3.5. *All the functors i^*, i_*, $i_!$, $j_!$, j^*, j_* constructed above are exact functors; and i_*, $j_!$, and j_* are fully faithful.*

3.5. **Proof of Theorem 3.3.** By construction $\mathrm{Ker}j^* = \{\binom{X}{Q}_\phi \in \underline{\Lambda\text{-}\mathcal{G}proj} \mid {}_BQ \text{ is projective}\}$. By Proposition 3.2 there is an exact sequence $0 \to M\otimes Q \xrightarrow{\phi} X \to \mathrm{Coker}\phi \to 0$ in Λ-$\mathcal{G}proj$. Since $M\otimes Q$ is a projective A-module, and hence an injective object in Λ-$\mathcal{G}proj$, it follows that ϕ splits and then $\binom{X}{Q}_\phi \cong \binom{M\otimes Q}{Q}_{\mathrm{Id}} \oplus \binom{X'}{0} = \binom{X'}{0}$ in $\underline{\Lambda\text{-}\mathcal{G}proj}$. Thus $\mathrm{Im}i_* = \mathrm{Ker}j^*$.

In the following $\binom{X}{Y}_\phi \in \underline{\Lambda\text{-}\mathcal{G}proj}$, $X' \in \underline{A\text{-}\mathcal{G}proj}$, and $Y' \in \underline{B\text{-}\mathcal{G}proj}$.

It is easy to see that a Λ-map $\binom{f}{0} : \binom{X}{Y}_\phi \to \binom{X'}{0}$ factors through a projective Λ-module if and only if the induced A-map $\mathrm{Coker}\phi \to X'$ factors through a projective A-module. This implies that the isomorphism (2.7) induces the following isomorphism, which are natural in both positions

$$\mathrm{Hom}_{\underline{\Lambda\text{-}\mathcal{G}proj}}(\binom{X}{Y}_\phi, \binom{X'}{0}) \cong \mathrm{Hom}_{\underline{A\text{-}\mathcal{G}proj}}(\mathrm{Coker}\,\phi, X'),$$

i.e., (i^*, i_*) is an adjoint pair.

It is easy to see that a Λ-map $\binom{f}{0} : \binom{X'}{0} \to \binom{X}{Y}_\phi$ factors through a projective Λ-module if and only if $f : X' \to X$ factors through a projective A-module. This implies that the isomorphism (2.8) induces the following isomorphism, which are natural in both positions

$$\operatorname{Hom}_{\underline{\Lambda\text{-}\mathcal{G}proj}}(\binom{X'}{0}, \binom{X}{Y}_\phi) \cong \operatorname{Hom}_{\underline{A\text{-}\mathcal{G}proj}}(X', X),$$

i.e., $(i_*, i^!)$ is an adjoint pair.

Note that $M \otimes Q$ is a projective A-module for any projective B-module Q. It is easy to see that a Λ-map $\binom{\phi(\mathrm{Id}_M \otimes g)}{g} : \binom{M\otimes Y'}{Y'}_{\mathrm{Id}} \to \binom{X}{Y}_\phi$ factors through a projective Λ-module if and only if $g : Y' \to Y$ factors through a projective B-module. This implies that the isomorphism (2.9) induces the following isomorphism, which are natural in both positions

$$\operatorname{Hom}_{\underline{\Lambda\text{-}\mathcal{G}proj}}(\binom{M\otimes Y'}{Y'}_{\mathrm{Id}}, \binom{X}{Y}_\phi) \cong \operatorname{Hom}_{\underline{B\text{-}\mathcal{G}proj}}(Y', Y),$$

i.e., $(j_!, j^*)$ is an adjoint pair.

Let $\binom{f}{g} : \binom{X}{Y}_\phi \to \binom{P'}{Y'}_\sigma$ be a Λ-map, $0 \to M \otimes Y' \xrightarrow{\sigma} P' \to \mathrm{Coker}\sigma \to 0$ an exact sequence with P' projective and $\mathrm{Coker}\sigma \in A\text{-}\mathcal{G}proj$. In the proof of Lemma 3.4 we know that $\binom{f}{g}$ factors through a projective Λ-module if and only if $g : Y \to Y'$ factors through a projective B-module. This implies that the map $\underline{g} \mapsto \underline{\binom{f}{g}}$ gives rise to the following isomorphism, which is natural in both positions

$$\operatorname{Hom}_{\underline{\Lambda\text{-}\mathcal{G}proj}}(\binom{X}{Y}_\phi, \binom{P'}{Y'}) \cong \operatorname{Hom}_{\underline{\Lambda\text{-}\mathcal{G}proj}}(Y, Y'),$$

i.e., (j^*, j_*) is an adjoint pair. Now the first assertion follows from Lemmas 3.5 and 1.3.

Assume that A and B are in additional finite-dimensional algebras over a field k. Note that Λ-$\mathcal{G}proj$ is a resolving subcategory of Λ-mod (see e.g. Theorem 2.5 in [Hol]). Since Λ is a Gorenstein algebra, it is well-known that Λ-$\mathcal{G}proj$ contravariantly finite in Λ-mod (see Theorem 11.5.1 in [EJ], where the result is stated for arbitrary Λ-modules, but the proof holds also for finitely generated modules. See also Theorem 2.10 in [Hol]). Then by Corollary 0.3 of H. Krause and Ø. Solberg [KS], which asserts that a resolving contravariantly finite subcategory in A-mod is also covariantly finite in A-mod, Λ-$\mathcal{G}proj$ is functorially finite in A-mod, and hence Λ-$\mathcal{G}proj$ has Auslander-Reiten sequences, by Theorem 2.4 of M. Auslander and S. O. Smalø [AS]. Since each distinguished triangle in the stable category $\underline{\mathcal{A}}$ of a Frobebius category $\mathcal{A}$ is induced by an exact sequence in $\mathcal{A}$, $\underline{\Lambda\text{-}\mathcal{G}proj}$ has Auslander-Reiten triangles. By assumption Λ is finite-dimensional k-algebra, thus $\underline{\Lambda\text{-}\mathcal{G}proj}$ is a Hom-finite k-linear Krull-Schmidt category, and hence by Theorem I.2.4 of I. Reiten and M. Van den Bergh [RV] $\underline{\Lambda\text{-}\mathcal{G}proj}$ has a Serre functor. Now the second assertion follows from Theorem 7 of P. Jørgensen [J], which claims that any recollement of a triangulated category with a Serre functor is symmetric. ■

3.6. By Theorem 3.3 we have

Corollary 3.6. *Let A be a Gorenstein algebra, and $T_2(A) = \begin{pmatrix} A & A \\ 0 & A \end{pmatrix}$. Then we have a recollement of triangulated categories*

$$\begin{tikzcd} \underline{A\text{-}\mathcal{G}proj} \arrow[r] & \underline{T_2(A)\text{-}\mathcal{G}proj} \arrow[l, shift right=2] \arrow[l, shift left=2] \arrow[r] & \underline{A\text{-}\mathcal{G}proj} \arrow[l, shift right=2] \arrow[l, shift left=2] \end{tikzcd} \ ;$$

and it is symmetric if A and B are finite-dimensional algebras over a field.

For the first part of Corollary 3.6 see also Theorem 3.8 in [IKM].

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